Indian J. Phys. **76A** (5), 473-476 (2002)

IJP A
an international journal

An appropriate candidate for exact distribution of closed random walks using quantum groups

S A Alavi* and M Sarbishaei
Department of Physics, Ferdowsi University of Mashhad, Mashhad, Iran-91735-654

E-mail : Alavi@karun.ipm.ac.ir

Received 12 April 2002, accepted 20 June 2002

Abstract We show that the structure of the quantum group $su_q(2)$ is intimately related to the random walks on a two dimensional lattice. Using this connection we obtain an appropriate candidate for the exact area distribution of closed random walks of length N on a two dimensional square lattice. We compare our results with exact enumeration.

Keywords Quantum groups, random walks

PACS Nos. 05.45.-a, 72.15.Qm, 72.10.Bg

1. Introduction

Let us consider a spinless electron on a two dimensional lattice and submitted to a uniform magnetic field along the z-direction and perpendicular to the plane of motion. The Hamiltonian of the system is

$$h = \frac{1}{2m}(\boldsymbol{p} + e\boldsymbol{A})^2 . \tag{1}$$

The system is not invariant under translations but there is an invariance under the so-called magnetic translation operators

$$w(\boldsymbol{a}) = \exp\left[\frac{i}{\hbar}\boldsymbol{a}.\boldsymbol{k}\right], \tag{2}$$

where $\boldsymbol{a}$ is an arbitrary two dimensional vector and $\boldsymbol{k}$ is

$$\boldsymbol{k} = k_x\hat{x} + k_y\hat{y} = \boldsymbol{p} + e\boldsymbol{A} + e\boldsymbol{r}\times\boldsymbol{B}. \tag{3}$$

There is a relationship between magnetic translation operators, quantum groups, Landau levels and the quantum Hall effect which has been the subject of study in several references [2-4]. In our previous work [5] we studied the application of quantum groups in gauge field theories.

It is shown that $w(a)$ satisfy the following relation

$$w(a)\,w(b) = \exp\left(i\frac{\gamma}{2}\boldsymbol{a}\times\boldsymbol{b}\right)w(a+b) \tag{4}$$

where $\boldsymbol{a}\times\boldsymbol{b} = a_1 b_2 - a_2 b_1$, which is exactly the q-commutation relation

$$w(a)\,w(b) = q\,w(b)\,w(a) . \tag{5}$$

Using (4) and (5) it is easy to show that the following combinations of the magnetic translation operators satisfy the $su_q(2)$ algebra :

$$j+ = \frac{w(\boldsymbol{a}) + w(\boldsymbol{b})}{q - q^{-1}} \tag{6}$$

$$j- = -\frac{w(-\boldsymbol{a}) + w(-\boldsymbol{b})}{q - q^{-1}} , \tag{7}$$

where $q^{j_3} = w(\boldsymbol{b}-\boldsymbol{a}) = j_z$ and $q = \exp\left(i\pi\frac{\Phi}{\Phi_0}\right)$. Φ and Φ_0 are the magnetic flux through the unit cell and the quantum of the flux respectively.

$$\Phi = \frac{1}{2}\boldsymbol{B}.(\boldsymbol{a}\times\boldsymbol{b}),\ \Phi_0 = \frac{\hbar}{e}. \tag{8}$$

* Corresponding Author

The $su_q(2)$ algebra is the q-deformation of the lie algebra $su(2)$ [6-9]. The generators of the $su_q(2)$ algebra satisfy the commutation relations

$$[j_3, j_\pm] = \pm j_\pm \,, \tag{9}$$

$$[j_+, j_-] = \frac{q^{2j_3} - q^{-2j_3}}{q - q^{-1}} \tag{10}$$

We observe that (4), (5) and the $su_q(2)$ algebra are invariant under following transformations

$$w(a) \to g(\gamma)\, w(a)\,, \tag{11}$$

$$w(-a) \to g^{-1}(\gamma)\, w(-a)\,, \tag{12}$$

or equivalently :

$$j_+ \to g(\gamma)\, j_+ \,, \tag{13}$$

$$j_- \to g^{-1}(\gamma)\, j_- \,, \tag{14}$$

$$j_z \to j_z \,. \tag{15}$$

2. Areas distribution of closed random walks

Bellissard *et al* [10] derived the distribution of areas of closed random walks using noncommutative geometry. The Harper model Hamiltonian [11] is given by

$$H = \sum_{|a|=1} w(\boldsymbol{a})\,. \tag{16}$$

We define the trace per unit area such that

$$T\big(w(\boldsymbol{m}_1)\, w(\boldsymbol{m}_2)\big) = \delta_{m_1+m_2,0}\,. \tag{17}$$

Using (16) and (17) it is easy to see that

$$T(H^N) = \sum_{\Gamma\ \text{Closed paths of length } N} e^{\frac{i\gamma}{2}A(\Gamma)}, \tag{18}$$

which is exactly the same as the one obtained in [10]. The sum is over the set of closed paths starting at the origin of length N. If Ω_N be the number of such closed paths, we have [10] :

$$\sum_{A=-A_{\max}}^{A_{\max}} P_N\left(\frac{A}{N}\right)\exp\left(\frac{ixA}{N}\right) = \Omega_N^{-1}\sum_{\Gamma}\exp\left(\frac{ixA(\Gamma)}{N}\right)$$

$$= \Omega_N^{-1}\, T(H^N)\big|_{\gamma=\frac{x}{N}}\,. \tag{19}$$

From (6), (7) and (16) we obtain

$$H = w(\boldsymbol{a}) + w(\boldsymbol{b}) + w(-\boldsymbol{a}) + w(-\boldsymbol{b}) = 2(q - q^{-1})J_y\,. \tag{20}$$

Thus, we observe that the Harper Hamiltonian operator is a generator of the $su_q(2)$ algebra. If the eigenvalue of j_3 is m, the eigenvalue of j_z will be $\pm e^{-\frac{m\gamma}{4}}$, with $q = \exp\left(-\frac{\gamma}{4}\right)$.

Using (17) or its equivalent,

$$T(w(c)) = \delta_{c,0}\,. \tag{21}$$

one can show that

$$T(w(\boldsymbol{b}-\boldsymbol{a})) = T\left(\sum w(\boldsymbol{a})\right) = T(j_y) = 0. \tag{22}$$

This result is also obtained from (9) and (10) directly. A straightforward calculation show that the eigenvalues of H have exponential form (see for example, eq. (18)). To obtain the eigenvalues of H, we note that

(i) E_m is a function of q-deformation parameter γ and the quantum number m. It also has exponential form and satisfies eq. (22).

From mathematical point of view, if a series of functions

$$\sum_{n=1}^{\infty} f_n(x) \tag{23}$$

converge for any x in some interval, then this series is finite and is a function of x, $f(x) = \sum_{n=1}^{\infty} f_n(x)$. This means that there is a closed expression for this series. The importance of a series (*e.g.* a solution of a given differential equation) lies in its convergence. For example, if ℓ be an integer, the series expansions of the solution of the Legendre equation $P_\ell(x)$ and $Q_\ell(x)$ converge and they convert to polynomials.

We know that the energy of a physical system must be finite, therefore the series (6) in [9] converge and there is a closed expression for the spectrum of H. On the other hand, we know that the operators j_y and j_z of $su(2)$ algebra do not commute and can not have common eigenvectors but their eigenvalues are the same. Therefore, there is no problem if the eigenvalues of the operators j_z and j_y of $su_q(2)$ algebra depend on the same quantum number m.

(ii) We know the eigenvalues of $\bar{H}$, in the limit of large N [10].

Therefore, we can write

$$E_m = \pm e^{u_m}(\gamma)\,. \tag{24}$$

We choose $u_m(\gamma) = -m\frac{\gamma}{4} + h(\gamma)$, then we have

$$E_m = \pm f(\gamma)\, e^{-m\frac{\gamma}{4}}, \tag{25}$$

where $f(\gamma) = \exp(h(\gamma))$. A more explicit calculation can be found in Appendix A. We can determine $f(\gamma)$ using $N \to \infty$ limit, then we will have

$$E_m = \pm 4\cos\left(\frac{m\gamma}{\sinh\left(\frac{N\gamma}{4}\right)}\right) \tag{26}$$

We have chosen $f(\gamma)$ such that $N \to \infty$ limit is correctly obtained :

$$f(\gamma) = 2\frac{\frac{\gamma}{4}}{\sinh\left(\frac{N\gamma}{4}\right)} + q\frac{\frac{\gamma}{4}}{\sinh\left(\frac{N\gamma}{4}\right)} \tag{27}$$

By introducing the new variable ℓ as

$$m = 2\ell + 1, \tag{28}$$

we have

$$E_\ell = \pm 4\cos\left(\frac{(2\ell+1)\gamma}{\sinh\left(\frac{N\gamma}{4}\right)}\right) \tag{29}$$

Expanding E_ℓ for small values of γ, $\gamma \to 0$, we obtain

$$E_\ell = \pm 4\left(1-(2\ell+1)\frac{\gamma}{4}+\left[(2\ell+1)^2+1\right]\frac{\gamma^2}{32}-O(\gamma^3)\right), \tag{30}$$

which are the eigenvalues used by Bellissard [10]. Now $T(H^N)$ is given by

$$T(H^N) = \sum\sum\left(E_\ell^{\pm}(\gamma)\right)^N\left(\frac{\gamma}{2\pi}\right) \tag{31}$$

where $\gamma = \frac{x}{N}$ is the multiplicity per unit area. This leads to

$$T(H^N) = \frac{4^{N+1}}{2\pi N}\cdot\frac{\frac{x}{4}}{\sinh\left(\frac{x}{4}\right)}\cos^N\left(\frac{\frac{x}{4}}{N\sinh\left(\frac{x}{4}\right)}\right), \tag{32}$$

for even N. This gives us the exact characteristic function for the distribution of area. In the limit of large N, we obtain

$$T(H^N(x)) = \frac{4^{N+1}}{2\pi N}\frac{\frac{x}{4}}{\sinh\left(\frac{x}{4}\right)}\left(1-\frac{1}{2N}\frac{\left(\frac{x}{4}\right)^2}{\sinh^2\left(\frac{x}{4}\right)}\right) + O\left(\frac{1}{N^2}\right) \tag{33}$$

which is also derived by Bellissard [10]. From (19) and (32), we have

$$\Omega_N = T(H^N)\Big|_{\gamma=0} = \frac{4^{N+1}}{2\pi N}\cos^N\left(\frac{1}{N}\right) \tag{34}$$

This gives the total number of closed paths of length N. In the limit of large N, we have

$$\Omega_N = \frac{4^{N+1}}{2\pi N}\left(1+\left(O\frac{1}{N}\right)\right). \tag{35}$$

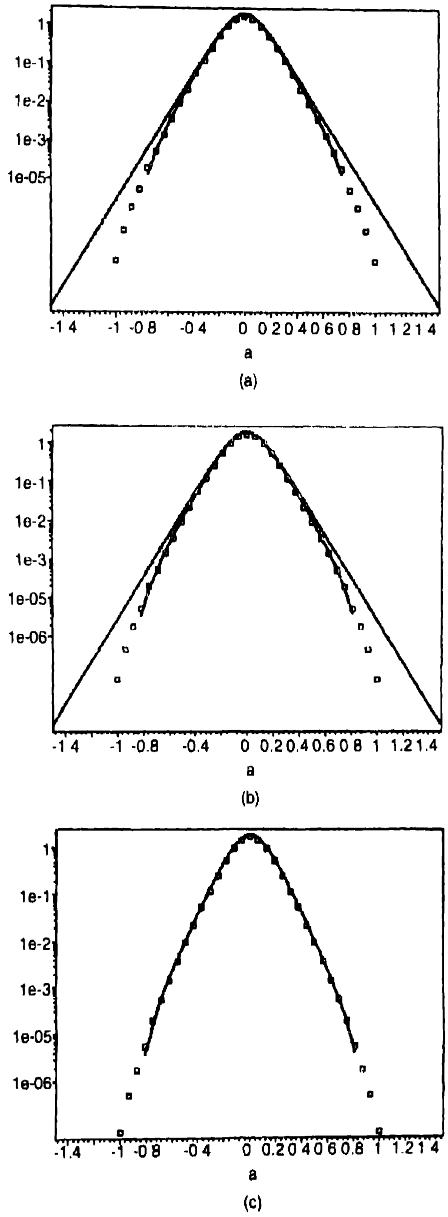

Figure 1. The exact and approximate distribution for areas of closed random paths. The exact distribution is plotted for $N = 16$. The solid points were obtained through exact enumerations [12].

Using (19), (32), (34) and the normalization condition, we can obtain a series expansion for the probability distribution :

$$P_N(a)=\frac{4}{\pi N}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\ell=0}^{m+1}$$

$$(-1)^{\ell}G_m^N\,2^{2m+1}\binom{2m+k}{k}\binom{2m+2}{2\ell}$$

$$\times\frac{(2m+2k+1)^{2m+2-2\ell}(16a^2)^{\ell}}{\left((2m+2k+1)^2+16a^2\right)^{2m+2}}\,\Gamma(2m+2), \tag{36}$$

where a is the renormalized area and A is the algebraic area.

We have plotted $P_N(a)$ as given by eq. (36) against exact enumeration [12] in Figure 1. In the summation of eq. (36) we have allowed m to range from 0 to 2 and k from 0 to 7000 for Figure 1 (a) and from 0 to 60000 for Figures 1(b) and (c). Comparing Figures 1 (a) and (b), we find that a better result will be obtained if we increase the range of summation over k. The same statement is also true for m. $m = 0$ and $m = 1$ give the eq. (9) in [10] and the $\frac{1}{N}$ correction term made by them respectively. Some of the G_m^N coefficients are presented in Appendix B.

Acknowledgment

One of the authors (S. A. A.) would like to thank Professor S. Rouhani for valuable discussions. He is also very grateful to Professor P. Kulish for his careful reading of the manuscript and for his valuable comments and finally I wish to thank K. Khabbazi and O. Naser Ghodsi for their assistance on computational aspects of this work.

References

[1] M Chaichian and A Demichev *Introduction to Quantum Groups* (Singapore : World Scientific) p183 (1996)

[2] Haru-Tada Sato *Mod Phys Lett* **A9** 1819 (1994)

[3] Haru-Tada Sato *Mod Phys Lett* **A9** 451 (1994)

[4] Alimohammadi and A Shafei Deh Abad *J Phys* **A29** 559 (1996)

[5] S A Alavi, M Sarbishaei and A Mokhtari *Indian J Phys.* **74A** (6) 589 (2000)

[6] P Kulish and Yu Reshtikhin *J. Sov Math* **23** 2435 (1983) (translation from Zapiski Nauch *Seminarov LOMI* **101** 101 (1981))

[7] G Drinfeld in *Proc ICM (Berkeley, CA)* ed A M Gleason (Providence, RI : AMS) P 798 (1986)

[8] M Jimbo *Lett. Math. Phys.* **10** 63 (1985) ; *Lett Math. Phys* **11** 247 (1986)

[9] M Chaichian and P Kulish *Phys. Lett.* **B 234** 72 (1990)

[10] Jean Bellissard *et al J. Phys* **A30** L707 (1997)

[11] P G Harper *Proc Phys Soc Lond.* **A68** 874 (1955) ; P G Harper *Proc. Phys. Soc. Lond* **A68** 879 (1955)

[12] R Afsari, N Sadeghi and S Rouhani *Exact Enumeration of Closed Random Path using TMS320 C6201* (Preprint No IS-00-21 Cond-mat/0102016)

Appendix A

The Harper Hamiltonian is

$$H=e^{a_1}+e^{a_2}+e^{-a_1}+e^{-a_2}=\sum e^{a_i}, \tag{A1}$$

where $\left[a_i, a_j\right]=u_{ij}(\gamma)$, $e^{a_i}e^{a_j}=e^{a_i+a_j}e^{\frac{1}{2}[a_i,\,a_j]}$ and

$$T\left(e^{a_i}\right)=\delta_{a_i,\,0}. \tag{A2}$$

then

$$H^N=\sum_{a_i}\sum_{a_j}\cdots\sum_{a_N}e^{a_i}e^{a_j}\cdots e^{a_N}. \tag{A3}$$

Therefore, we have

$$T(H^N)=\sum_m e^{u_m(\gamma)}, \tag{A4}$$

$u_m(\gamma)$ must be determined using boundary conditions

Appendix B

The first four G_m^N coefficients are as follows .

$$G_0^N=1, \tag{B1}$$

$$G_1^N=\frac{1}{2N} \tag{B2}$$

$$G_2^N=\frac{1}{24}\frac{1}{N^3}+\frac{1}{8}\frac{N-1}{N^3}, \tag{B3}$$

$$G_3^N=\frac{1}{720}\frac{1}{N^5}-\frac{1}{48}\frac{N-1}{N^5}-\frac{1}{48}\frac{(N-1)(N-2)}{N^5}. \tag{B4}$$